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Quinquisection of the Cylotomic Equation.

BY J. C. GLASHAN, *Ottawa, Canada.*

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Let p be a prime number $= 5n + 1$, t be a primitive root of p and

$$(\chi^p - 1)/(\chi - 1) = 0, \qquad \text{(i)}$$

also let

$$\left.\begin{aligned} \eta_0 &= \chi + \chi^{t^5} + \chi^{t^{10}} + \chi^{t^{15}} + \cdots, \\ \eta_1 &= \chi^{t} + \chi^{t^6} + \chi^{t^{11}} + \chi^{t^{16}} + \cdots, \\ \eta_2 &= \chi^{t^2} + \chi^{t^7} + \chi^{t^{12}} + \chi^{t^{17}} + \cdots, \\ \eta_3 &= \chi^{t^3} + \chi^{t^8} + \chi^{t^{13}} + \chi^{t^{18}} + \cdots, \\ \eta_4 &= \chi^{t^4} + \chi^{t^9} + \chi^{t^{14}} + \chi^{t^{19}} + \cdots \end{aligned}\right\} \qquad \text{(ii)}$$

and

$$\left.\begin{aligned} \eta_0\eta_1 &= a\eta_0 + \gamma\eta_1 + b\eta_2 + c\eta_3 + d\eta_4, \\ \eta_0\eta_2 &= \beta\eta_0 + e\eta_1 + \delta\eta_2 + f\eta_3 + g\eta_4, \end{aligned}\right\} \qquad \text{(iii)}$$

$$\therefore \quad \eta_0 + \eta_1 + \eta_2 + \eta_3 + \eta_4 + 1 = 0. \qquad \text{(iv)}$$

Making $\chi = 1$ in equations (iii) and dividing through by n,

$$\left.\begin{aligned} a + \gamma + b + c + d &= n, \\ \beta + e + \delta + f + g &= n. \end{aligned}\right\} \qquad \text{(v)}$$

On substituting χ^{t}, χ^{t^2}, χ^{t^3}, χ^{t^4} successively for χ, we obtain the eight equations which may be formed from (iii) by the substitutions $(\eta_0\eta_1\eta_2\eta_3\eta_4)^m$, $m = 1, 2, 3, 4$. These, with the two equations of (iii), form the group of ten equations

$$\left.\begin{aligned} \eta_m\eta_{m+1} &= a\eta_m + \gamma\eta_{m+1} + b\eta_{m+2} + c\eta_{m+3} + d\eta_{m+4}, \\ \eta_m\eta_{m+2} &= \beta\eta_m + e\eta_{m+1} + \delta\eta_{m+2} + f\eta_{m+3} + g\eta_{m+4}, \\ &\text{in which} \\ \eta_{m+5} &= \eta_m \\ &\text{and} \\ m &= 0,\ 1,\ 2,\ 3,\ 4. \end{aligned}\right\} \qquad \text{(vi)}$$

From (vi) and (iv),

$$\left.\begin{aligned}(\eta_0\eta_1)\,\eta_2 &= -b\eta_2 + (\alpha - b)\,\eta_0\eta_2 + (\gamma - b)\,\eta_1\eta_2 + (c - b)\,\eta_2\eta_3 + (d - b)\,\eta_2\eta_4\,,\\ = (\eta_1\eta_2)\,\eta_0 &= -d\eta_0 + (\alpha - d)\,\eta_0\eta_1 + (\gamma - d)\,\eta_0\eta_2 + (b - d)\,\eta_0\eta_3 + (c - d)\,\eta_0\eta_4\,,\\ = (\eta_0\eta_2)\,\eta_1 &= -e\eta_1 + (\beta - e)\,\eta_0\eta_1 + (\delta - e)\,\eta_1\eta_2 + (f - e)\,\eta_1\eta_3 + (g - e)\,\eta_1\eta_4\,.\end{aligned}\right\}\quad\text{(vii)}$$

On making $\chi = 1$ in the right-hand members of these equations, we obtain

$$\begin{aligned}(\alpha + \gamma + c + d - 4b)\,n^2 - bn &= (\alpha + \gamma + b + c - 4d)\,n^2 - dn\\ &= (\beta + \delta + f + g - 4e)\,n^2 - en\,;\\ \therefore\quad (n - 5b)\,n - b &= (n - 5d)\,n - d = (n - 5e)\,n - e\,;\\ \therefore\quad b &= d = e\,. \end{aligned}\qquad\text{(viii)}$$

Similarly, from $(\eta_0\eta_1)\,\eta_3 = (\eta_1\eta_3)\,\eta_0 = (\eta_0\eta_3)\,\eta_1$, we obtain

$$c = f = g\,. \qquad\text{(ix)}$$

On writing b for both d and e, and c for both f and g, and substituting for the products $\eta_0\eta_1$, $\eta_0\eta_2$, …. , $\eta_2\eta_4$ in the right-hand members of the equations (vii) the values of these products as given in (vi), we obtain

$$b = (\alpha - b)(\alpha - \beta) + (\gamma - b)(\beta - b) + (c - b)(\gamma - c)\,. \qquad\text{(x)}$$

Similarly, from $(\eta_0\eta_2)\,\eta_4 = (\eta_2\eta_4)\,\eta_0 = (\eta_0\eta_4)\,\eta_2$, we obtain

$$c = (\beta - c)(\beta - \gamma) + (\delta - c)(\gamma - c) + (b - c)(\delta - b)\,. \qquad\text{(xi)}$$

It is worthy of notice that (xi) may be obtained from (x) by the substitution $\{(\alpha\beta\gamma\delta)(bc)\}$. This is a consequence of the substitution $\{(\eta_1\eta_2\eta_4\eta_3)(\alpha\beta\gamma\delta)(bc)\}$, leaving the equation-group (vi) unchanged.

The eight equations of (v), (viii), (ix), (x) and (xi) may be written under the form

$$\left.\begin{aligned}&b = d = e,\\ &c = f = g,\\ &\alpha + \gamma = n - 2b - c,\\ &\beta + \delta = n - b - 2c,\\ &\alpha\gamma + \beta\delta = n^2 - (4n + 1)(b + c) + 5b^2 + 7bc + 5c^2,\\ &5\alpha^2\gamma^2 - \alpha\gamma\,[5\,\{n^2 - n\,(5b + 3c) + 7b^2 + 7bc + 3c^2\} - 6b - 4c]\\ &\qquad + n^4 - 5n^3\,(2b + c) - n^2\,(40b^2 + 35bc + 10c^2 - 2b - c)\\ &\qquad - n\,(75b^3 + 90b^2c + 40bc^2 + 10c^3 - 11b^2 - 7bc - 2c^2)\\ &\qquad + 5\,(11b^4 + 17b^3c + 9b^2c^2 + 3bc^3 + c^4)\\ &\qquad - (16b^3 + 15b^2c + 3bc^2 + c^3) + 2b^2 = 0\,.\end{aligned}\right\}\quad\text{(xii)}$$

The last of these equations may also be written under the form

$$4pA^2 = (B + 5p)^2 - 5D^2$$

in which

$$A = 25(b + c) - 2(p + 1),$$
$$B = 25\{(b + c)^2 + 5bc - (p + 1)(b + c)\} + p^2 + p + 1,$$
$$D = 50\alpha\gamma - (p - 1)^2 + 5(p - 1)(5b + 3c) - 25(7b^2 + 7bc + 3c^2) + 10(3b + 2c).$$

Computing the values of the symmetric functions $\Sigma\eta_0\eta_1$, $\Sigma\eta_0\eta_1\eta_2$, etc., in terms of the coefficients α, β, γ, δ, b, c, the quintic whose roots are η_0, η_1, η_2, η_3, η_4 is found to be

$$\begin{aligned}\eta^5 + \eta^4 - 2n\eta^3 - \{p(b + c) - 2n^2\}\eta^2 \\ + [p\{n(b + c) - b^2 - 3bc - c^2\} - n^3]\eta \\ + (2b^2 - \alpha\gamma)(2c^2 - \beta\delta) - (b + c)^4 + n(b + c)^3 + bc(b + c)^2 \\ - (3n + 1)bc(b + c) + b^2c^2 = 0. \qquad \text{(xiv)}\end{aligned}$$

If K denote the constant term of this equation

$$\begin{aligned}5K = p\{(b - c)\alpha\gamma - n^2(2b + c) + n(2b + c)(3b + c) \\ - (6b^3 + 10b^2c + 8bc^2 + c^3) + b^2\} + n^4. \qquad \text{(xv)}\end{aligned}$$

If (xiv) be transformed by substituting $\frac{1}{5}(z - 1)$ for η, it becomes

$$\begin{aligned}z^5 - 10pz^3 - 5p\{25(b + c) - 2(p + 1)\}z^2 \\ - 5p[25\{(b + c)^2 + 5bc - (p + 1)(b + c) + p^2 + p + 1]z \\ + 3125\{(2b^2 - \alpha\gamma)(2c^2 - \beta\delta) - (b + c)^4\} \\ + 125\{5(b + c)^2 + 2bc\}(b + c)(p - 1) \\ + 125(5bc + p)\{5(b + c)^2 + 5bc - p(b + c)\} + 5p(p^2 - p + 1) + 1 = 0. \qquad \text{(xvi)}\end{aligned}$$

Writing this equation

$$z^5 - 10pz^3 - 5pAz^2 - 5pBz - pC = 0$$

it will be found that

$$\left.\begin{aligned}4(B + p) = A^2 - 125(b - c)^2, \\ \{2C - A(B - p)\}^2 = 125(b - c)^2\{(B + 5p)^2 - 4pA^2\}.\end{aligned}\right\} \qquad \text{(xvii)}$$

To solve equation (xiv), let

$$(\theta^5 - 1)/(\theta - 1) = 0,$$

and $$\eta_m = \tfrac{1}{5}(y_0 + y_1\theta^{-m} + y_2\theta^{-2m} + y_3\theta^{2m} + y_4\theta^m),$$

and $\therefore$ $$y_m = \eta_0 + \eta_1\theta^m + \eta_2\theta^{2m} + \eta_3\theta^{3m} + \eta_4\theta^{4m},$$

$$m = 0,\ 1,\ 2,\ 3,\ 4$$

then will

$$\left.\begin{aligned} &y_0 = -1, \\ &y_1 y_4 = y_2 y_3 = p, \\ &y_1^2 y_3 + y_2^2 y_1 + y_3^2 y_4 + y_4^2 y_2 = pA, \\ &y_1^3 y_2 + y_2^3 y_4 + y_3^3 y_1 + y_4^3 y_3 = p(B+p), \\ &y_1^5 + y_2^5 + y_3^5 + y_4^5 = pC, \end{aligned}\right\} \tag{xviii}$$

$$\left.\begin{aligned} \therefore\ &y_1^2 y_3 + y_4^2 y_2 = \tfrac{1}{2}p\{A + 5(b-c)\sqrt{5}\} = \tfrac{1}{2}px, \\ &y_2^2 y_1 + y_3^2 y_4 = \tfrac{1}{2}p\{A - 5(b-c)\sqrt{5}\} = \tfrac{1}{2}p\lambda, \\ &y_1^2 y_3 - y_4^2 y_2 = \tfrac{1}{2}p\sqrt{(x^2 - 16p)} = \tfrac{1}{2}p\mu, \\ &y_3^2 y_4 - y_2^2 y_1 = \tfrac{1}{2}p\sqrt{(\lambda^2 - 16p)} = \tfrac{1}{2}p\nu, \end{aligned}\right\} \tag{xix}$$

$$\left.\begin{aligned} \therefore\ &y_1 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p(x+\mu)^2(\lambda-\nu)\}}, \\ &y_2 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p(x-\mu)(\lambda-\nu)^2\}}, \\ &y_3 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p(x+\mu)(\lambda+\nu)^2\}}, \\ &y_4 = \tfrac{1}{2}\sqrt[5]{\{\tfrac{1}{2}p(x-\mu)^2(\lambda+\nu)\}}, \end{aligned}\right\} \tag{xx}$$

and

$$\eta_m = \tfrac{1}{5}(y_0 + y_1\theta^{-m} + y_2\theta^{-2m} + y_3\theta^{2m} + y_4\theta^{m}). \tag{xxi}$$

In the Proceedings of the London Mathematical Society, vol. XII, p. 16, the late Professor Cayley has given tables of the values of the coefficients α, β, γ, δ, b, c, d, e, f, g and of the coefficients of the quintic in η for all values of the prime $p = 5n + 1$ up to $p = 71$, and on page 63 of vol. XVI of the Proceedings he has given from Legendre's "Théorie des Nombres," 3e edition, t. II, p. 213, the quintic in η for $p = 641$.* See also Cayley's Collected Mathematical Papers, vol. XI, p. 316, and vol. XII, p. 73. The following tables extend these values to $p = 641$. The values in table 1 are subject to the substitutions $\{(\alpha\beta\gamma\delta)(bgdf)(ce)\}^m$.

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*In Cayley's Table 2, in the equation for $p = 31$, coefficient of η^2, for -2 read -21, and in the equation for $p = 61$, coefficient of η^0, for -23 read -13. In the equation from Legendre, for $+5238$ read $+5328$.

TABLE I.

VALUES OF

p	a β	γ e	b δ	c f	d g	p	a β	γ e	b δ	c f	d g
11	1	0	0	1	0	281	14	10	9	14	9
	0	0	0	1	1		8	9	11	14	14
31	1	0	2	1	2	311	10	17	12	11	12
	2	2	0	1	1		14	12	14	11	11
41	2	2	1	2	1	331	14	9	14	15	14
	3	1	0	2	2		10	14	12	15	15
61	3	2	2	3	2	401	15	18	16	15	16
	0	2	4	3	3		22	16	12	15	15
71	2	4	3	2	3	421	19	18	14	19	14
	5	3	2	2	2		20	14	12	19	19
101	3	6	4	3	4	431	20	14	16	20	16
	6	4	4	3	3		12	16	18	20	20
131	5	4	6	5	6	461	16	18	21	16	21
	2	6	8	5	5		24	21	15	16	16
151	8	6	4	8	4	491	18	21	20	19	20
	6	4	4	8	8		26	20	14	19	19
181	8	6	7	8	7	521	24	18	19	24	19
	3	7	10	8	8		15	19	22	24	24
191	8	4	9	8	9	541	18	28	21	20	21
	8	9	5	8	8		23	21	24	20	20
211	6	8	11	6	11	571	24	26	20	24	20
	9	11	10	6	6		18	20	28	24	24
241	12	8	8	12	8	601	23	30	22	23	22
	6	8	10	12	12		24	22	28	23	23
251	8	10	12	8	12	631	26	30	23	24	23
	14	12	8	8	8		25	23	30	24	24
271	10	9	12	11	12	641	28	24	24	28	24
	6	12	14	11	11		18	24	30	28	28

TABLE II.

COEFFICIENTS OF THE QUINTIC EQUATIONS.

p	η^5	η^4	η^3	η^2	η^1	η^0
11	1	1	-4	-3	$+3$	$+1$
31	1	1	-12	-21	$+1$	$+5$
41	1	1	-16	$+5$	$+21$	-9
61	1	1	-24	-17	$+41$	-13
71	1	1	-28	$+37$	$+25$	-1
101	1	1	-40	$+93$	-21	-17
131	1	1	-52	-89	$+109$	$+193$
151	1	1	-60	-12	$+784$	$+128$
181	1	1	-72	-123	$+223$	-49
191	1	1	-76	-359	-437	-155
211	1	1	-84	-59	$+1661$	$+269$
241	1	1	-96	-212	$+1232$	$+512$
251	1	1	-100	-20	$+1504$	$+1024$
271	1	1	-108	-401	-13	$+845$
281	1	1	-112	-191	$+2257$	$+967$
311	1	1	-124	$+535$	-413	-539
331	1	1	-132	-887	-1843	-1027
401	1	1	-160	$+369$	$+879$	-29
421	1	1	-168	$+219$	$+3853$	-3517
431	1	1	-172	-724	$+1824$	$+1728$
461	1	1	-184	-129	$+4551$	$+5419$
491	1	1	-196	$+59$	$+2019$	$+1377$
521	1	1	-208	-771	$+4143$	$+2083$
541	1	1	-216	$+1147$	-805	-2629
571	1	1	-228	$+868$	$+3056$	-7552
601	1	1	-240	$+1755$	-3731	$+2399$
631	1	1	-252	$+2095$	-5785	$+5069$
641	1	1	-256	-564	$+5328$	-5120
	a	$5b$	$10c$	$10d$	$5e$	

$$af - 3be + 2cd \equiv 0 \text{ mod. } p,$$
$$ae - 4bd + 3c^2 \equiv 0 \quad \text{``}$$
$$bf - 4ce + 3d^2 \equiv 0 \quad \text{``}$$